CALIFORNIA INSTITUTE OF TECHNOLOGY
Division of the Humanities and Social Sciences
Pasadena, California 91125

SOME NEW IMPOSSIBILITY THEOREMS *

John A. Ferejohn
and
David M. Grether

*Forthcoming Public Choice, Winter 1977

Social Science Working Paper
Number 124
Revised July 1976

SOME NEW IMPOSSIBILITY THEOREMS

John A. Ferejohn
and
David M. Grether

California Institute of Technology
Pasadena, California

Mas-Collel and Sonnenschein [1972] have proved a well-known impossibility theorem for acyclic social preference. They showed that if social preference is acyclic for each configuration of individual preferences, and if the mapping between individual and social preferences satisfies the Pareto axiom, the independence of irrelevant alternatives, and May's positive responsiveness axiom, then there must be an individual i such that for any x and y, $xP_iy \implies xRy$. This individual might be considered a "vetoer."

Bordes [1974] and Sen [1975] have noted, that Mas-Collel and Sonnenschein actually proved a stronger result than the one they stated. One way of stating this result is as follows: any mapping that takes each configuration of individual preferences into a complete, reflexive social preference relation that has no strict preference cycles of length three in such a way that the Pareto axiom, May's positive responsiveness axiom, and the independence of irrelevant alternatives are satisfied must necessarily have the property that a vetoer exists. In other words the requirement that social preference be acyclic is not necessary to obtain impossibility results of the sort reported by Mas-Collel and Sonnenschein.

While the Mas-Collel Sonnenschein theorem is familiar to most scholars in the field, it is considered by some to be of limited significance due to its use of May's positive responsiveness axiom. The authors themselves remark that "As stated, the condition is very strong and perhaps somewhat unappealing" [1972, p. 101]. Indeed it is. The positive responsiveness condition requires that if for a given preference configuration society is indifferent between a pair of alternatives, then if one person in his preferences raises one of the alternatives relative to the other and everyone else's preferences with respect to that pair remain the same, society switches to strict preference. Mas-Collel and Sonnenschein go on to remark that PR is employed to exclude the admissibility of what they call "degenerate" social preference functions, e.g. unanimity rules.

In this paper we produce a class of impossibility results that are related in spirit to those obtained by Mas-Collel and Sonnenschein. In particular we demonstrate that there is an inescapable conflict between consistent social preference or choice and the requirement that no individuals in society be "too" powerful. In section I we show that if social choice is acyclic, it cannot satisfy some relatively weak conditions designed to ensure that power not be too narrowly held. In section II we introduce a class of axioms restricting the social choice function and show that various conditions prohibiting the concentration of power cannot be simultaneously satisfied. Finally, in section III we show that a Mas-Collel Sonnenschein type theorem is available even when the rationality condition is greatly weakened.

Society is a set of n individuals having a preference relation R_i which is a weak order on X, the finite set of possible social states. The social choice from a set V where V is a subset of X is denoted $C(V, R_1, R_2, \ldots, R_n)$ or C(V) where the dependence on the preference of the individuals is unambiguous. C(V) is assumed to be nonempty and contained in V. We utilize the following standard definitions.

P(Pareto Principle): $\forall i \quad xP_iy \Longrightarrow \{x\} = C(\{x,y\})$.

I(Independence of Irrelevant Alternatives): Let $(R_1,\dots,R_n)$ and $(R_1',\dots,R_n')$ be two sets of individual preferences. $\forall V \subseteq X$ if $\forall x,y \in V$, $\forall i \; xR_iy \Longleftrightarrow xR_i'y$, then $C(V,R_1,\dots,R_n) = C)V,R_1',\dots,R_n')$.

N(Neutrality):

$$\forall x,y,z,w \in X\text{: } [(\forall i: xR_iy \Longleftrightarrow zR_i'w) \text{ and } (\forall i: yR_ix \Longleftrightarrow wR_i'z)] \Longrightarrow$$
$$[(x \in C(\{x,y\}, R_1,\dots,R_n) \Longleftrightarrow z \in C(\{z,w\}, R_1',\dots,R_2')) \text{ and}$$
$$(y \in C(\{x,y\}, R_1,R_2,\dots,R_n) \Longleftrightarrow w \in C(\{z,w\}, R_1',\dots,R_n'))].$$

We shall (somewhat loosely) say that a choice function is acyclic if and only if it has an acyclic rationalization, that is $\forall V \subseteq X$, $\forall(R_1,\dots,R_n)$, $\exists R \in \mathcal{A}$ such that $\emptyset \neq C(V,R_1,\dots,R_n) = \{x : x \in V, xRy, \forall y \in V\}$ where $\mathcal{A}$ is the set of acyclic relations on X.

I. A SIMPLE IMPOSSIBILITY THEOREM

In this section we propose a new axiom that excludes "degenerate" social preference functions and then prove a new impossibility theorem. Unlike the Mas-Collel Sonnenschein theorem which requires only that social choice be acyclic on a triple of alternatives, we shall use the full strength of the acyclicity condition. On the other hand, we shall be able to obtain our impossibility results without the imposition of the independence of irrelevant alternatives axiom.

<u>Axiom 1(k):</u> $\forall x,y \in X$ and for each subset c of individuals such that $|c| \leq k$ where k is a positive integer.

$$(\forall i \in c,\ xP_iy \text{ and } \forall j \notin c,\ yP_jx) \Longrightarrow \{y\} = C(\{x,y\}).$$

We may now prove the following theorem:

<u>Theorem 1</u>: If $|X| \geq \frac{1}{k}n \geq 3$, there is no choice function C( ) satisfying axiom 1(k) and acyclicity.

<u>Proof</u>: There exist integers ℓ and r such that $n = \ell \cdot k + r$, $0 \leq r < k$. Society may be partitioned into J subsets c_j such that $|c_j| \leq k$ where

$$J = \ell \text{ if } r = 0$$
$$= \ell + 1 \text{ if } r > 0.$$

Suppose we have the following preference configuration:

c_1	c_2	c_3	$\dots$	c_J
x_1	x_2	x_3	$\dots$	x_J
x_2	x_3	x_4	$\dots$	x_1
$\vdots$	$\vdots$	$\vdots$		
x_J	x_1	x_2	$\dots$	x_{J-1}

Then $\{x_1\} = C(\{x_1,x_2\})$, $\{x_2\} = C(\{x_2,x_3\}),\dots,\{x_{J-1}\} = C(\{x_{J-1},x_J\})$ and $\{x_J\} = C(\{x_J,x_1\})$.

The requirement that $C(\{x_1,\dots,X_J\})$ be nonempty contradicts the assumption that C( ) is acyclic.

Q.E.D.

Note that for $k = 1$ the axiom we are using is a slight strengthening of the Mas-Collel Sonnenschein requirement that there be no weak dictator-D. They formulate their axiom as follows:

NWD-D: $\forall i$, for some $x, y \in X$, (xP_iy and yP_jx, $\forall j \neq i$), and $\{y\} = C(\{x, y\})$.

In fact we have the following corollary to Theorem 1.

<u>Corollary</u>: If $|X| \geq n \geq 3$, there is no choice function satisfying NWD-D, neutrality and acyclicity.

<u>Proof</u>: NWD-D says that for each i there is an $x, y \in X$ such that (xP_iy and yP_jx, $j \neq i$) and $\{y\} = C(\{x, y\})$. Neutrality ensures that Axiom 1(1) holds, thus satisfying all the conditions of Theorem 1.

Q. E. D.

A number of similar results have appeared in the literature. For example, theorem 1, which is closely related to results reported in Ferejohn and Grether [1974], shows that as the requirement that power not be narrowly held is strengthened, the number of alternatives society must have before it in order to obtain an impossibility theorem is reduced.

II. IMPOSSIBILITY RESULTS FOR NONRATIONAL CHOICE FUNCTIONS

Following work by Sen [1971], Ferejohn and Grether [1975], and others, we report here some impossibility theorems for social choice functions which do not satisfy a rationality condition. Numerous authors have produced axiomatizations of rationality. We will produce a class of axioms that are similar in spirit to ones introduced by Sen

[1971]. Sen [1975] proved that the following two axioms together are equivalent to rationality (acyclicity).

$\alpha 2$: $x \in C(V) \Longrightarrow x \in C(\{x, y\})$, $\forall y \in V$,

$\gamma 2$: $x \in C(\{x, y\})$, $\forall y \in V \Longrightarrow x \in C(V)$.

It is easy to show that if C satisfies $\alpha 2$, then the relation $\overline{R}$ defined as $x\overline{R}y \Longleftrightarrow x \in C(\{x, y\})$ is acyclic. Indeed, the following axiom is sufficient for the acyclicity of $\overline{R}$:

α-: $\forall V \subseteq X$, $\exists x \in C(V)$ such that $x \in C(\{x, y\})$, $\forall y \in V$.

Clearly, if $\overline{R}$ is interpreted as the social preference relation, we can obtain strengthenings of many previously proved impossibility theorems. For example, Sen Showed that if a social choice procedure satisfies the Pareto axiom, independence of irrelevant alternatives, May's positive responsiveness, and axiom α-, then there must be a vetoer. Natural strengthenings of our results in section I may be obtained straightforwardly.

Here we introduce another set of axioms that relate naturally to Sen's α-.

$\alpha(k)$: $\forall V \subseteq X$ such that $|V| \leq k$. $\exists x \in C(V)$ such that $x \in C(\{x, y\})$, $\forall y \in V$.

Notice that $\alpha\text{-} = \bigcup_k \alpha(k)$ and $\alpha(k) \Longrightarrow \alpha(k-1)$. We can also demonstrate the following result.

<u>Lemma 1</u>: If C satisfies $\alpha(k)$, then $\overline{R}$ has no strict preference cycles of length k or shorter.

Proof: Suppose there is a k-cycle over the elements $\{x_1, x_2, \ldots, x_n\}$. Then $\{x_1\} = C(\{x_1, x_2\})$, $\{x_2\} = C(\{x_2, x_3\}), \ldots, \{x_{k-1}\} = C(\{x_{k-1}, x_k\})$ and $\{x_k\} = C(\{x_k, x_1\})$. Notice that there is no element z in $C(\{x_1, x_2, \ldots, x_k\})$ with the property that $z \in C(\{x_i, z\})$, $\forall i = 1, \ldots, k$. Thus $\alpha(k)$ requires $C(\{x_1, \ldots, x_n\})$ to be empty.

Q. E. D.

We note that $\alpha(3)$ is the α-- axiom introduced by Sen and that it is the weakest axiom in the family that we are studying.

Our primary interest now is in obtaining a class of theorems which indicate the relation between this nested family of consistency axioms and the family of axioms governing the distribution of power introduced in section I. In particular we will show the following.

Theorem 2: if $|X| > \frac{1}{\ell}n$, then if C satisfies $\alpha(k)$, it cannot satisfy axiom $1(\ell)$ when $n = (k - 1)\ell + p$ and $p < \ell$.

Proof: Suppose axiom $1(\ell)$ is satisfied. Then, using the construction in the proof of theorem 1, we can find a configuration for which there is a k-cycle.

Q. E. D.

Theorem 2 indicates exactly how increasing the strength of a rationality requirement (i. e. letting k increase) allows us to obtain stronger impossibility results (in the sense that axiom $1(\ell)$ becomes less restrictive as ℓ decreases).

III. AN IMPOSSIBILITY THEOREM FOR NONBINARY CHOICE

In common with most theorems in the literature the results given above depend upon the behavior of the social choice function on the two-element subsets of X. In this section we investigate the possibility of obtaining related results when the social choice funceion does not operate on such small subsets.

It will be noted that the essential feature of Sen's α type axioms is that at least some of the elements chosen in a large set be chosen as well in certain small subsets. In particular, $\alpha 2$, α-, and $\alpha(k)$ require that the small subsets in question be the pairs. Here we require that some element that is chosen in a large set is chosen in all ℓ-element subsets of that set. We propose the following axiom:

$A(\ell)$: For each $T \subseteq X$, $|T| \geq \ell$, $\exists x \in C(T)$ such that $x \in C(S)$, $\forall S \subseteq T$ such that $|S| = \ell$.

It will be seen that we do not restrict the behavior of the choice function on the small sets and in fact do not require that the choice function even be defined on sets with fewer than ℓ elements.

We will show that a choice function cannot satisfy $A(\ell)$ and a fairly natural nonbinary axiom requiring that power not be held too narrowly. We propose the following axiom requiring that choice be weakly democratic.

$WD(k, \ell)$: $\forall x, y \in X$ and $\forall c$, such that $|c| \leq k$, if xP_iy, $\forall i \in c$ and yP_jx, $\forall j \notin c$, then $\forall S \subseteq X$, $|S| \geq \ell$, $y \in S \Longrightarrow x \notin C(S)$.

This axiom requires that for large sets (those with at least ℓ elements), coalitions with fewer than k people cannot force a desired element into the choice set.

We may now establish the following theorem:

Theorem 3: If $k\ell \leq n$ and $|X| \geq \frac{1}{k}n$, C cannot satisfy $A(\ell)$ and $WD(k, \ell)$.

Proof: Suppose $k\ell \leq n$. Then $n = mk + r$, $0 \leq r < k$ and we can divide society into M subsets $|c_i| \leq k$, $i = 1,2,\dots,M$, where $M = m$, $r = 0$, or $M = m + 1$, $r > 0$, $M \geq \ell$. Consider the following configuration:

$$\begin{array}{ccccc}
c_1 & c_2 & c_3 & \cdots & c_M \\
x_1 & x_2 & x_3 & \cdots & x_M \\
x_2 & x_3 & x_4 & \cdots & x_1 \\
\cdot & \cdot & \cdot & & x_2 \\
\cdot & \cdot & \cdot & & \vdots \\
\cdot & \cdot & \cdot & & \\
x_M & x_1 & x_2 & & x_{M-1}
\end{array}$$

Note that since $|\{x_1, x_2, \dots, x_M\}| \geq \ell$, $A(\ell)$ requires that $x_j \in C(\{x_1, x_2, \dots, x_M\})$ and that x_j be contained in all ℓ-element subsets of $\{x_1, x_2, \dots, x_M\}$. Consider an ℓ-element subset, S, containing x_j and x_{j-1}. Note that $x_j P_i x_{j-1} \Longleftrightarrow i \in c_j$ and $x_{j-1} P_i x_j$, $\forall i \notin c_j$. Thus $x_j \notin C(S)$ by $WD(k, \ell)$.

Q. E. D.

It is of some interest to note that the structure of the proof of theorem 3 is identical to that of theorems 1 and 2, in spite of the fact that theorem 3 is established in a nonbinary setup. The reason for this is that $A(\ell)$ is sufficient to ensure that the following binary relations have no P^{ℓ}-cycles of length greater than $\ell - 1$. $xR^{\ell}y \Longleftrightarrow \exists S \subseteq X$, $|S| = \ell$ with $x \in C(S)$, $y \in S$. It should be noted that R^2 is simply Sen's "base relation," and that his α- axiom guarantees its acyclicity. For completeness we give the following example that indicates that while $A(\ell)$ precludes P^{ℓ}-cycles longer than $\ell - 1$, it does not rule out cycles of length $\ell - 1$.

Example: $X = \{x_1, x_2, x_3, w, z\}$

$$\begin{aligned}
\{w, z\} &= C(\{x_1, x_2, x_3, w, z\}) \\
\{w\} &= C(\{x_1, x_2, x_3, w\}) \\
\{z\} &= C(\{x_1, x_2, x_3, z\}) \\
\{x_1, w\} &= C(\{x_1, x_2, w, z\}) \\
\{x_2, w\} &= C(\{x_2, x_3, w, z\}) \\
\{x_3, w\} &= C(\{x_1, x_3, w, z\})
\end{aligned}$$

Note that $A(4)$ is satisfied and that $x_1 P^4 x_2, x_2 P^4 x_3, x_3 P^4 x_1$.

IV. DISCUSSION

We think that the results given here help clarify the relationship between the consistency of social choice and the requirement that the distribution of power in society should not be too unequal. Of course it is this relationship that is at the core of the theory of social choice, but the classical results of Arrow [1963], Sen [1971], and Brown [1974] have demonstrated impossibility results in only a small fraction of cases. Furthermore the consistency axioms employed in these works are quite strong. In this paper we have examined a number of consistency conditions, weaker than those found in the classical papers, and a range of axioms governing the distribution of power. It was found, not surprisingly in view of earlier results, that the more egalitarian society is required to be, the less consistent it can be.

The theorems of Arrow, Sen, and Brown tell us exactly what the power structure of society must look like if we require that social preference be a weak order, quasi-transitive, or acyclic, respectively. We have given analogous results for social choice functions that satisfy a weaker family of consistency requirements. In this case the power structure is not necessarily oligarchic but, as in the case of Sen's and Brown's theorems, either some small group is vested with power or else social choice is indecisive.

REFERENCES

Arrow, Kenneth J. Social Choice and Individual Values, 2nd ed. New York: John Wiley and Sons, 1963.

Bordes, G. "Alpha Rationality and Social Choice: A General Possibility Theorem." Mimeographed. Laboratoire d'Analyse et de Recherche Economiques, University de Bordeaux I (1974).

Brown, D. J. "An Approximate Solution to Arrow's Problem." Journal of Economic Theory, 9 (1974).

Ferejohn, John A., and Grether, David M. "On a Class of Rational Social Decision Procedures." Journal of Economic Theory, 8 (August 1974), 471-482.

__________, and __________. "Weak Path Independence." Social Science Working Paper No. 80, California Institute of Technology. (Forthcoming, Journal of Economic Theory.)

Guilbaud, T. "The Theories of the General Interest and the Logical Problem of Aggregation," in Readings in Mathematical Social Science, (English translation). Edited by Lazarfeld and Henry. Chicago: Science Research Associates, Inc. (1966).

Mas-Collel, A., and Sonnenschein, H. "General Possibility Theorems for Group Decisions." Review of Economic Studies, 39 (1972).

Sen, A. K. "Choice Functions and Revealed Preference." Review of Economic Studies, 38 (1971).

__________. "Social Choice Theory: A Re-examination." Paper presented at the Third World Congress, Econometric Society, Toronto, August 1975. Forthcoming in Frontiers in Quantitative Economics. Edited by M. D. Intriligator and D. Kendrick. Amsterdam: North-Holland; and forthcoming in Econometrica.